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Comments on Summation of Slowly Converging Series

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the same normalized velocity-distribution function, $f(u, v, w)$, for the three velocity components, u, v, and w, then a well-known relation connects ω and β.[1] This may be written

$$\frac{1}{\omega_p^2}=\int_{-\infty}^{+\infty}\frac{f_z(w)dw}{(\omega-\beta w)^2}, \tag{1}$$

where

$$\omega_p^2=\eta\frac{\rho_0}{\epsilon_0} \quad\text{and}\quad f_z(w=)\int_{-\infty}^{+\infty}du\int_{-\infty}^{+\infty}dvf(u,v,w);$$

η is the charge to mass ratio of the electron and ϵ_0 the permittivity of free space. At the 1949 Institute of Radio Engineers Conference on Electron Devices the writer described a simple method which may give information about the roots β of (1) for real ω. The method will be outlined here since it does not seem to be generally known.

In (1) put $\omega/w=t$ and $f_z(\omega/t)=g(t)$, so that $g(t)$ is normalized to unity if f_z is. Note that for real ω, if β is a root of (1), so is β^*, its complex conjugate. Consider then the equation satisfied by β^*, which is

$$\int_{-\infty}^{+\infty}\frac{g(t)dt}{(t-\beta^*)^2}-\frac{\omega^2}{\omega_p^2}=0.$$

Integrating by parts, one has

$$\int_{-\infty}^{+\infty}\frac{(dg/dt)dt}{t-\beta^*}-\left[\frac{g(t)}{t-\beta^*}\right]-\frac{\omega^2}{\omega_p^2}=0, \tag{2}$$

where the term in square brackets indicates the sum of the values of the enclosed term evaluated (with the signs appropriate to the integration) at all the finite discontinuities of $g(t)$, say $t_1, t_2, \cdots, t_n$. Inspection shows that the three terms of (2) may be interpreted successively as the complex electric field (E_x+jE_y) at the point $\beta=x+jy$ because of a surface distribution of line charges of density dg/dt at $x=t$; lumped charges $\pm g(t_r)$ at $x=t_1, t_2, \cdots, t_n$ and a uniform field $-\omega^2/\omega_p^2$ in the x direction; y may be assumed positive. If, then, $\beta=x+jy$ is a neutral point of this electrostatic system, β^* is a root of (2). It should be remarked that the total charge on $y=0$ is zero and the field due to the charges must then vanish at large distances.

As an example of the use of this analog consider a distribution which is zero except for $0<t_1<t<t_2$ and let it be flat or monotone or have a single maximum in this interval. For such distributions the corresponding charge distribution, including any lumped charges, consists of a single positive region and a single negative region with the former lying entirely to the left of the latter. The electric vector at β is now easily seen to lie to the left of a line joining β to the point of separation of positive and negative charge and thus its component normal to this line is always in the same direction as that of the component of the field, $-\omega^2/\omega_p^2$. There can therefore be no neutral point for $y\neq 0$. It follows that a minimum in $g(t)$ or $f_z(w)$ is a necessary condition for a complex root of (2). By considering E_x along $y=0$ it is clear that a pair of roots exists for real β, one on each side of the charge distribution.

Consider now a case in which $f_z(w)$ is symmetrical about a maximum at $w=0$. Any three-dimensional distribution which is symmetrical in u, v, and w will lead to such an $f_z(w)$. For this case $g(t)$ is monotone decreasing (increasing) for t negative (positive) if f runs from $w=-\infty$ to $w=+\infty$; if the distribution is finite, lying between $w=-w_0$ and $w=w_0$, $g(t)$ follows the above rule, except in the range $-\omega/w_0<t<\omega/w_0$, where it is zero. By symmetry the field E_y vanishes generally only on $y=0$; in addition it is zero for the finite distribution on $y=0$ for $-\omega/w_0<x<\omega/w_0$. Along $x=0$ the field E_x is positive and increases as y decreases. If $E_x(0,0)>\omega^2/\omega_p^2$ or $\omega_p^2>(\omega^2/E_x(0,0))$, there will be a purely imaginary root (cut off). If $E_x(0,0)<\omega^2/\omega_p^2$ there can be no root at all in the unbounded case, but for the finite distribution, since $E_x(x,0)$ increases from $E_x(0,0)$ to infinity as $|x|$ goes from 0 to ω/v_0, there will be a pair of symmetrically disposed roots in this range.

In this analog, single velocity beams are represented by dipoles, as may easily be seen if they are considered to be the limit of narrow rectangular distributions, since the latter are to be represented simply as a pair of lumped charges at the ends of the distribution. The existence of double stream gain for sufficiently strong pairs of dipoles and the behavior of the gain as a function of separation and so forth are readily discussed qualitatively.

[1] See, for example, D. Bohm and E. P. Gross, Phys. Rev. 75, 1851 (1949).

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IN a letter[1] to the editor published in this Journal in August, 1953, Gumowski develops a formula for the rapid summation of slowly converging series. We wish to point out that although his formula is independently derived, it is not a really new result, being, in essence, equivalent to the process of adding up the first few terms of the series and applying Euler's summation formula adapted to a converging series to the succeeding terms.

Euler's formula[2] is

$$f_0+f_1+\cdots+f_s=\int_0^s f(x)dx+\tfrac{1}{2}(f_s+f_0)+\sum_{k=1}^{\lambda}\frac{B_{2k}}{(2k)!}D^{2k-1}(f_s-f_0)+R_\lambda \tag{1}$$

where

$$R_\lambda=\int_0^s P_{2\lambda+1}(x)f^{(2\lambda+1)}(x)dx,$$

$$P_{2\lambda+1}(x)=(-1)^{\lambda-1}\sum_{i=1}^{\infty}\frac{2\sin 2i\pi x}{(2i\pi)^{2\lambda+1}},$$

and B_{2k} are the Bernoulli numbers. For convergent series we have that $\lim_{s\to\infty}(D^m f_s)=0$ for m, a non-negative integer. In Eq. (1) set $f_i=a_{n+i}$, $p'(n)=a_n$ and let both s and λ "go to infinity." In making these substitutions note that

$$\int_0^\infty f(x)dx=\int_0^\infty a_{n+x}dx=\int_n^\infty a_x dx.$$

We obtain for *convergent* series

$$\sum_{i=0}^{\infty}a_{n+i}=\int_n^\infty a_x dx+\tfrac{1}{2}a_n-\sum_{k=1}^{\infty}\frac{B_{2k}}{(2k)!}D^{2k}p(n)+R_\infty, \tag{1'}$$

where

$$R_\infty=\lim_{\lambda\to\infty}\int_n^\infty P_{2\lambda+1}(x)p^{(2\lambda+2)}(x)dx;$$

but

$$\lim_{\lambda\to\infty}P_{2\lambda+1}(x)=\lim_{\lambda\to\infty}(-1)^{\lambda-1}\sum_{i=1}^{\infty}\frac{2\sin 2i\pi x}{(2i\pi)^{2\lambda+1}}=0,$$

and since also $\lim_{\lambda\to\infty}p^{(2\lambda+2)}(x)$ is finite for $0<n\leq x\leq\infty$ (for convergent series) we have that $R_\infty=0$. Thus, substituting $R_\infty=0$ in (1′) and substituting the result in the identity

$$\sum_{k=1}^{\infty}a_k=\sum_{k=1}^{n}a_k+\sum_{i=0}^{\infty}a_{n+i}-a_n,$$

we obtain

$$\sum_{k=1}^{\infty}a_k=\sum_{k=1}^{n}a_k+\int_n^\infty a_x dx-\tfrac{1}{2}a_n-\sum_{k=1}^{\infty}\frac{B_{2k}}{(2k)!}D^{2k}p(n). \tag{2}$$

This is Gumowski's formula with some changes in notation. From the definition of b_m in the cited letter[1] it follows that $b_m=B_m/m!$ for $m>1$. Further, $B_3=B_5=B_7=\cdots=0$. The equivalence of (2) with Gumowski's formula is now clear. (The usage of $a_x dx$ instead of $a_n dn$ avoids a slight ambiguity, *viz.*—the lower limit on the integral is a chosen constant, n, whereas the "n's" in $a_n dn$ are obviously not meant to be constant.)

[1] I. Gumowski, J. Appl. Phys. 24, 1068 (1953).
[2] See, for example, K. Knopp, *Theory and Application of Infinite Series* (Blackie and Son, Ltd., Glasgow, 1951), p. 524.