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Probability and Statistical Models for Racing

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Abstract

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KEYWORDS: favorite-longshot bias, ordering probability, running time distribution, horse race

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1. Introduction

Racing data provides a rich source of analysis for quantitative researchers to study multi-entry competitions. In particular, horse racing has been well studied by researchers in multiple disciplines; including economists, psychologists, management scientists, statisticians, probability theorists, as well as professional gamblers, see Hausch et al (1994a) which covers articles from all these areas. We will focus on horse race data in this paper but the methodologies proposed are transferable to other types of racing such as car, dog, and boat racing.

We study two areas in this paper. Firstly, a favorite-longshot bias is often found in gambling data. The general interpretation is that since the reward from a longshot (if it wins) is higher than that from a favorite, gamblers tend to underbet favorites and overbet longshots. See Ali (1977), Snyder (1978), Asch et al (1982), Ziemba and Hausch (1987), and Lo (1994a), which all concluded the presence of this bias in US data with the exception of Busche and Hall (1988) using Hong Kong data. We apply a model proposed by Lo (1994a) and Bacon-Shone, Lo & Busche (1992a) to investigate the favorite-longshot betting bias using horse race data across the world. The result shows that the bias phenomenon is not universal, possibly due to difference in pool size. Economic interpretation using utility theory will also be provided. It is important to note that this bias is also reported in other areas, e.g. Ziemba (2004). While we focus on win bets here, more complex bets have also been studied elsewhere, e.g. Lo and Busche (1994).

Our second area is predicting higher order probabilities such as the probabilities of finishing second and third. The procedure of estimating ordering probabilities typically is: 1) knowledge of winning probabilities (i.e. finishing first); 2) estimating the mean running times using winning probabilities; and 3) estimating ordering probabilities using the mean running times. Various probability distributions have been proposed to model running time. The first model proposed by Harville (1973) is a simple way of computing ordering probabilities based on winning probabilities, and can be derived assuming that the running times are independent exponential or extreme-value. Henery (1981) and Stern (1990) proposed to use normal and gamma distributions respectively for running times. However, both the Henery and Stern models are complicated to apply in practice. Bacon-Shone, Lo & Busche (1992b) and Lo and Bacon-Shone (1994) showed that the Henery and Stern models fit better than the Harville model for particular racing data. Additionally, Lo and Bacon-Shone (2008) proposed a simple practical approximation for both the Henery and Stern models. We extend Henery's independent normal distribution assumption to include certain correlation and variance structure and apply the extended model to real data.

2. Study of Favorite-Longshot Bias

2.1 Model and Results

We examine whether gamblers tend to underbet favorites and overbet longshots in order to aim at a higher reward if the longshot wins. Researchers using US horse race data consistently concluded the presence of this bias. However, Busche and Hall (1988) did not see such a bias using data from Hong Kong racetracks. We study whether this bias phenomenon holds for multiple racetracks from different countries.

While previous researchers used variety of methods to study the favorite-longshot bias, we apply a more rigorous but simple statistical model proposed by Lo (1994a) and Bacon-Shone, Lo & Busche (1992a). Define:

P_i = Bet fraction (or % of win bet) on horse i, i.e. consensus win probability, $i = 1, \ldots, n$

= (1- track take)/(1 + O_i), where O_i = Win odds on i, and track take is a percentage from the total betting pool to cover taxes, expenses, and profits,

π_i = objective (true) win probability of i. Then,

$$\pi_i = \frac{P_i^\beta}{\sum_{j=1}^{n} P_j^\beta} \qquad (1)$$

The interpretation of the parameter β is straightforward:

$\beta>1 \rightarrow$ risk-prefer,

$\beta=1 \rightarrow$ risk-neutral,

$\beta<1 \rightarrow$ risk-averse.

Table 1 shows the results when applying model (1) to multiple racetracks.

Table 1: International Comparison of Favorite-Longshot Bias

Racetrack	# races	Estimated β	p-value for H1: β not equal to 1	Average pool size
US (Quandt's 83-84):				
Atlantic City	712	1.10	0.08	unknown
Meadowlands	705	1.12	0.02	$52K
US (Ali's 70-74):				
Saratoga	9,072	1.16	~0	$25K
Roosevelt	5,806	1.13	~0	$218K
Yonkers	5,369	1.13	~0	$228K
Japan (90)	1,607	1.07	0.01	$168K
Hong Kong (81-89):				
Happy Valley	2,212	1.04	0.25	$1.1M
Shatin	1,943	0.94	0.04	$1.1M
China (23-35):				
Shanghai	730	1.03	0.38	unknown

In Table 1, the first column indicates various racetracks in the US, Japan, Hong Kong, and Mainland China, the second column shows the number of races at each track, the third column shows the estimated parameter β followed by the p-value associated with H_0: $\beta = 1$ versus H_1: $\beta \neq 1$ in the next column. It can be seen that the β's are significantly different from (in fact, greater than) 1, indicating a longshot-favorite bias, for all racetracks in the US and Japan but not for Hong Kong and Shanghai racetracks. The last column indicates the average size of the winning pool for each racetrack, showing a huge difference between Hong Kong and the rest of the racetracks. One hypothesis is that because of the much higher pool size in Hong Kong, the higher expected gain has attracted more careful research work done in the area, resulting in more accurate bets. For example, Benter (1994) reports on some scientific research conducted by a betting syndicate in Hong Kong.

2.2 Utility Interpretation

Next, we employ economic utility theory to study the favorite-longshot bias based on model (1). Assuming expected utility maximizer is indifferent between betting on any horses in a race, see Ali (1977), it can be shown that:

Expected utility of $i = E(U_i) = \pi_i U(1+O_i) = K \; \forall i$,
then

$$U(1+O_i) = K/\pi_i = K\left[1+\frac{\sum_{j\neq i} P_j^{\beta}}{(1-t)^{\beta}}(1+O_i)^{\beta}\right] \propto (1+O_i)^{\beta}$$, a power function

where t = track take.

Then, the Arrow – Pratt Measure of Absolute Risk Aversion

$$= \frac{-U''(x)}{U'(x)} = -(\beta-1)/x \qquad (2)$$

< 0, and increases with wealth, if $\beta > 1$.

The negative Arrow-Pratt Measure means that bettors take more risk as capital decline, i.e. "Risk-lovers," if $\beta > 1$. See Ali (1977) and LeRoy & Werner (2006).

3. Predicting Ordering Probabilities with Running-time Distribution

3.1 Overview

While predicting the winner is important, it is also important to predict second and third places. In horse racing, this is related to exacta and trifecta bets. To estimated ordering probabilities such as π_{ij} (probability of i finishing first and j finishing second) and π_{ijk} (probability of i finishing first, j finishing second, and k finishing third), Harville (1973) proposed the following simple formulas:

$$\pi_{ij} = \frac{\pi_i \pi_j}{1-\pi_i}, \qquad (3)$$

$$\pi_{ijk} = \frac{\pi_i \pi_j \pi_k}{(1-\pi_i)(1-\pi_i-\pi_j)} \qquad (4)$$

where π_i can be estimated by bet fraction P_i.

(3) and (4) can be derived assuming independent exponential running times (or equivalently in this context, extreme-value), a simple and perhaps unrealistic assumption.

Other running time distributions for racing data have been proposed to estimate ordering probabilities. However, the formulas for ordering probabilities are usually not as simple as (3) and (4). Let T_i be the running time of horse *i,* then the following procedure can be used to estimate ordering probabilities:

Step 1 : Estimate π_{i}. This can be estimated by the bet fraction P_i.

Step 2 : Solve the following equation to estimate θ_{i} :

$$\pi_i = P(T_i < \underset{r \neq i}{MIN}\{T_r\}) = \int_{-\infty}^{\infty} \prod_{r \neq i} [1 - F(t_i \mid \theta_r)] f(t_i \mid \theta_i) dt_i, \qquad (3)$$

where $\theta_i = E(T_i)$ or location parameter,

and $f(.)$ and $F(.)$ are pdf and cdf, resp.

Step 3 : Estimate π_{ij} :

$$\pi_{ij} = P(T_i < T_j < \underset{r \neq i,j}{MIN}\{T_r\}) = \int_{-\infty}^{\infty} F(t_j \mid \theta_i) \prod_{r \neq i,j} [1 - F(t_j \mid \theta_r)] f(t_j \mid \theta_j) dt_j \qquad (4)$$

Similar integrals can be computed for higher order probabilities.

Henery (1981) assumed that $T_i \sim N(\theta_i, 1)$ independently. This will involve solving the system of integral equations in (3) and (4) using numerical integrations and is not practical to use in real races. Similar practical difficulties apply to the gamma model proposed by Stern (1990), where an extra shape parameter is involved. Lo and Bacon-Shone (2008) proposed a simple approximation to both the Henery and Stern models:

$$\pi_{ijk} = \pi_i \frac{\pi_j^{\lambda}}{\sum_{s \neq i} \pi_s^{\lambda}} \frac{\pi_k^{\tau}}{\sum_{t \neq i,j} \pi_t^{\tau}} \qquad (7)$$

where π_i's can be estimated by bet fractions P_i's,
λ and τ are parameter values in Lo and Bacon−Shone(2007).
Note that for exponential time, $\lambda = \tau = 1$,
(7) reduces to (5) and (6).

Lo and Bacon-Shone (1994) found that the Harville model had a systematic bias in estimating ordering probabilities based on Hong Kong data and the Henery model was clearly superior in terms of model fit. Bacon-Shone, Lo, and Busche (1992b) had a similar conclusion using Meadowlands data, however, Lo (1994b) found that the Stern model with shape parameter = 4 was better than both Henery and Harville using Japan data. All these models and approximations are based on the assumption of independent running times. We will now relax this assumption in a generalization of the Henery model.

3.2 Extension of the Henery Model

Recall that Henery (1981) assumed that $T_i \sim N(\theta_i,1)$ independently. A natural extension is to assume a constant correlation, i.e. Corr(T_i ,T_i) = ρ for all i and j (and all races). However, it can be easily shown that this is equivalent to the Henery model where running times are independent so a more complex structure is proposed:

a) Non - constant correlation : $\rho_{ij} = \psi_i \psi_j \ \forall i \neq j$, (8)

where $\log(\frac{\psi_i}{1-\psi_i}) = -\delta - \gamma(\theta_i - \bar{\theta}),\ \bar{\theta} = \frac{1}{n}\sum_i \theta_i$, (9)

i.e. correlations tend to be higher for stronger pairs .

b) Non - constant variance : $\sigma_i = \exp[\kappa(\theta_i - \bar{\theta})]$, (10)

i.e. if $\kappa > 0$, weaker horses will have higher variance.

If $\gamma = \kappa = 0$, it reduces to Henery.

To estimate the parameters δ, γ, and κ in (8) – (10) using maximum likelihood, we choose the top 5 finishing positions for constructing the likelihood function as the correlation and non-constant variance structure is expected to show higher degree of importance in estimating higher order probabilities. Following Steps 1 – 3 in Section 3.1 for models (8) – (10), and using a first order Taylor series approximation similar to Henery (1981)'s, it can be shown that with Steps 1 and 2:

$$\theta_i \approx \frac{\phi(z_0)(n-1)(z_i - z_0)}{M_1} \qquad (11)$$

where $z_0 = \Phi^{-1}(1/n), z_i = \Phi^{-1}(P_i),$

P_i = win bet fraction for horse i,

$M_1 = A' + B'\mu_{1;n} - A'\mu_{1;n}^{(2)},$

$\mu_{1;n}$ = expected value of minimum standard normal order statistic in a race of n horses,

$\mu_{1;n}^{(2)}$ = second moment about origin of minimum standard normal order statistic in a race of n horses,

ϕ and Φ are standard normal pdf and cdf, resp.,

$$A' = -\kappa n' - \frac{\gamma e^{\delta} n'}{\delta'(\delta'^2 - 1)},\ B' = \frac{1}{\sqrt{1 - \frac{1}{\delta'^2}}},$$

$\delta' = 1 + e^{\delta}$, and $n' = 1 - 1/n.$

And, with Step 3, to predict horses *i1, i2, …, i5* finishing the first 5 positions:

$$P(T_{i1} < \ldots < T_{i5} < \min_{r \neq i1,\ldots,i5}\{T_r\}) = \pi_{i1,i2,i3,i4,i5}$$

$$\approx \frac{\Phi\{C_5 + \nu_5[\sum_{r=1}^{5}\theta_{ir}M_r + \sum_{r=1}^{5}\theta_{ir}\sum_{r=1}^{5}M_r/(n-5)]\}}{\sum_{t1,t2,t3,t4,t5}\Phi\{C_5 + \nu_5[\sum_{r=1}^{5}\theta_{tr}M_r + \sum_{r=1}^{5}\theta_{tr}\sum_{r=1}^{5}M_r/(n-5)]\}} \qquad (12)$$

where

$C_5 = \Phi^{-1}(1/{}_nP_5), \nu_5 = \frac{1}{\phi(C_5)\,{}_nP_5}, {}_nP_5 = n(n-1)\ldots(n-5+1),$

$M_i = A' + B'\mu_{i;n} - A'\mu_{i;n}^{(2)},$

$\mu_{i;n}$ = ith expected standard normal order statistic in a sample of n,

$\mu_{i;n}^{(2)}$ = ith second moment about origin of minimum standard normal order statistic in a sample of n,

and the denominator is summed over all permutations of horses finishing in the first 5 positions.

Appendix A outlines the proof for (11) and (12). It can be easily shown that the log likelihood of data from multiple races is:

$$\log \text{lik} = \sum_{l=1}^{\# races} \log \pi_{[12345],l}$$

where $\pi_{[12345],l}$ is the probability of the top 5 horses actually finishing in the first 5 positions in race l, as a function of the parameters to be estimated.

The above models have been fit on 400 8-horse races in Hong Kong. The model objective is to predict the probabilities of horses finishing in the first 5 positions.

Table 2: Comparison Between Henery and Extended Models

Model	Estimates	p-value of likelihood ratio test relative to Henery
a1) Non-constant correlation (γ only)	$\gamma = 0.58$	0.06
a2) Non-constant correlation (γ and δ)	$\gamma = 0.60$, δ=0.05	0.18
b) Non-constant variance	$\kappa = 0.08$	0.06

(Note: p-value above indicates the significance of the difference between the extended model and the original Henery model by the likelihood ratio test.)

Table 2 indicates that the non-constant correlation structure with slope γ only or non-constant variance structure shows some promise (significant at 6% level).

Improving the ordering probability estimates is only meaningful if they can be used in practice. Hausch, Ziemba and Rubinstein (1981) assumed the Harville (1973) model and developed a Kelly criterion (Breiman (1960), Algoet and Cover (1988), Haigh (2000)) based stochastic nonlinear programming model to optimize bets. Using a similar optimization algorithm, Lo, Bacon-Shone and Busche (1995) demonstrated the superiority of using the Henery and Stern models in terms of long-term returns in different racetracks. Hausch, Lo, and Ziemba (1994b), however, concluded that the Harville model was slightly better than the Henery model using a small data set in a particular type of bets. For future research, it will be interesting to see whether the above non-constant correlation or non-constant variance structure, while marginally significantly better in terms of fit, will demonstrate a better result in betting. Further, it will be better if a simpler approximation similar to (7) can be derived for (8) – (10) to be applied in practice.

4. Conclusion

Racing data is so rich that it provides many opportunities for academia and practitioners to study. While this paper focused on horse-racing data, the techniques can be applied to other types of racing such as cars, boats, and dogs.

In this paper, we studied two research areas in racing data. First, based on a rigorous yet simple statistical model, we discovered that the so-called favorite long-shot bias is not a universally true phenomenon although it appears to be consistent in the US. We suggested a hypothesis to explain the results but racetrack data from more countries can be used for further research. Second, we attempted to improve existing ordering probability models using more complex correlation and variance structures. The result shows some promise and deserves further investigation especially in terms of generating returns in racetrack betting.

Appendix A: Approximation Formulas for the Non-Constant Correlation and Non-Constant Variance Structures

This appendix provides an outline of the proof for (11) and (12), which are a first-order Taylor series approximation to the solution to (8) – (10). It is a similar approach used by Henery (1981).

Consider the structures in (8) – (10), it can be shown that the running times among horses in the same race can be expressed as (see Johnson and Kotz (1972, p.47))

$$\frac{T_i - \theta_i}{\sigma_i} = \psi_i U_0 + \sqrt{1-\psi_i^2}\,U_i,\ \ i = 1,2,...,n \qquad (A.1)$$

where $U_0, U_1, ..., U_n \overset{iid}{\sim} N(0,1)$.

This means $T_1, ..., T_n$ are correlated with each other only through U_0, which also implies :

$T_i \mid u_0 \sim N(\theta_i + \psi_i u_0 \sigma_i, \sigma_i^2 (1-\psi_i^2))$ independently, (A.2)

where

$\psi_{\mathrm{i}} = \{1 + \exp[\delta + \gamma(\theta_i - \overline{\theta})]\}^{-1}$ and $\sigma_i = \exp[\kappa(\theta_i - \overline{\theta})]$, according to (9) and (10).

Then,

$$\begin{aligned}\pi_i &= P(T_i < \min_{r \neq i}\{T_r\}) \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \prod_{r\neq i}[1 - F_r(y \mid u_0;\theta_r)]f_i(y \mid u_0;\theta_i)\,dy\,\phi(u_0)\,du_0 \\ &= \Phi\{g_i(\theta_1,...,\theta_n)\} \qquad (A.3),\end{aligned}$$

where $F_i(.)$ and $f_i(.)$ above are the cdf and pdf of T_i given u_0.

Applying the first order Taylor series approximation to $g(.)$ in (A.3) around θ_i's = 0:

$$g_i(\theta_1,...,\theta_n) \approx g_i(0,...,0) + \sum_{j=1}^{n}\theta_j\left[\frac{\partial g_i(\theta_1,...,\theta_n)}{\partial\theta_j}\right|(\theta_1,...,\theta_n) = (0,...,0)],$$

(11) can be obtained with the following additional approximations:

$\frac{\partial\psi_r}{\partial\theta_i} \approx 0$ and $\frac{\partial\sigma_r}{\partial\theta_i} \approx 0$ for $r \neq i$, and the assumption that $\sum_{j=1}^{\mathrm{n}}\theta_j = 0$.

We are now going to find a general approximate formula for $\pi_{i1,i2,...,im}$, (m=1,…n).

$$\begin{aligned}\pi_{i1,i2,...,im} &= P(T_{i1} < ... < T_{im} < \min_{r\neq i1,...im}\{T_r\}) \\ &= \sum_{A_{m+1},...,A_n} P(T_{i1} < ... < T_{im} < A_{m+1} < ... < A_n) = \Phi\{h(\theta_1,...,\theta_n)\}, \qquad (A.4)\end{aligned}$$

where $A_{m+1},...,A_n$ is a permutation of $T_{i,m+1},...,T_{in}$ and the summation is taken over all possible permutations.

Each term in the above summation can be evaluated as follows:

$$P(T_{i1} < ... < T_{im} < A_{m+1} < ... < A_n)$$
$$= \int_{-\infty}^{\infty} \phi(u_0) \int_{-\infty}^{\infty} f_{i1}(t_{i1} \mid u_0) \int_{t_{i1}}^{\infty} f_{i2}(t_{i2} \mid u_0) \, ... \int_{a_{n-1}}^{\infty} f_n(a_n \mid u_0)\, da_n \, ... dt_{i2} dt_{i1} du_0 \quad ,$$

where $f_{i1}(.),..., f_{im}(.), f_{m+1}(.),..., f_n(.)$ are the pdf's of $T_{i1},...,T_{im}, A_{m+1},..., A_n$, respectively.

Using the first order Taylor series approximation again to h(.) in (A.4) around θ_i's = 0, the numerator of (12) can be obtained for m=5. The denominator of (12) is there to make sure that the following is satisfied:

$$\sum_{i1} ... \sum_{i5} \pi_{i1,...,i5} = 1.$$

It can be shown that, at least with the approximation in (11) and (12), when $\delta = 0$, (9) and (10) are equivalent, and thus either γ or κ is needed but not both.

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